Direct and Related Far-Field Inverse Scattering Problems for Spherical Electromagnetic Waves in Chiral Media

N.M.Berketis *

Independent Researcher, Athens, Greece

Abstract

This paper studies the direct and inverse scattering problem when the incident electromagnetic field is a time harmonic point – generated wave in a chiral media and the scatterer is a perfectly conducting sphere. The exact Green’s function and the electric far-field patterns of the scattering problem are constructed. For a small sphere, a closed-form approximation of the scattered wave field at the source of the incident spherical wave is obtained. Also treats the same inverse problem using far-field results via the leading order term in the low-frequency asymptotic expansion of the scattering cross-section.

Introduction

In a homogeneous isotropic chiral media the electromagnetic fields are composed of left – circularly polarized (LCP) and right – circularly polarized (RCP) components, which have different wave numbers and independent directions of propagation.

The LCP and RCP components are assumed to be spherical Beltrami fields since in practice such wave fields are more readily realized.

In this work, the author has studied the electromagnetic waves in chiral media produced by a point source in the vicinity of the scatterer. In particular, in [1], [2], reciprocity, optical and general scattering theorems for stimulation of point-source asymmetric media have been demonstrated. This paper studies the inverse problem of far field [2]. Specifically, we measure the scattering cross-section for a five-point source area.

In the second Section, considering Bohren decomposition into suitable Beltrami fields, we formulate the direct scattering problem of a spherical electromagnetic wave by a perfectly conducting obstacle. This problem is well posed, the existence and uniqueness has been proved in [1], [3].

In the third Section, after expanding the incident field in terms of spherical wave functions, we obtain the exact solution of the scattering problem as well as an expansion for the electric far-field pattern [2].

Finally in the fourth Section, we consider either LCP or RCP incidence and we obtain an approximation of the scattering cross-section. For the far-field experiments, we measure the scattering cross-section for various point source locations [2].

Statement of the problem

Our goal is to study the direct and inverse scattering problems when the incident electromagnetic field is a time harmonic point – generated wave in a chiral medium and the scatterer is a perfectly conducting sphere of radius a centered at the origin. The exterior space $(r=|\mathbf{r}|>a)$ is an infinite homogeneous isotropic chiral medium with chirality measure β, electric permittivity ε and magnetic permeability μ.

We consider a time harmonic spherical electromagnetic wave due to a point source at P_0 with position vector $\mathbf{r}_0$ with respect to an origin O in the vicinity of the scatterer. In order to define spherical electromagnetic fields $\mathbf{E}_{\mathbf{r}_0}$, $\mathbf{H}_{\mathbf{r}_0}$, we make use of the Bohren decomposition into Beltrami fields $\mathbf{Q}_{L,\mathbf{r}_0}$ and $\mathbf{Q}_{R,\mathbf{r}_0}$, as follows

$$\begin{aligned} \mathbf{E}_{\mathbf{r}_0} &= \mathbf{Q}_{L,\mathbf{r}_0} + \mathbf{Q}_{R,\mathbf{r}_0} \\ \mathbf{H}_{\mathbf{r}_0} &= \frac{1}{i\eta}(\mathbf{Q}_{L,\mathbf{r}_0} - \mathbf{Q}_{R,\mathbf{r}_0}) \end{aligned} \tag{1}$$

Where $\eta=(\mu/\varepsilon)^{1/2}$ is the intrinsic impedance of the chiral medium. The Beltrami fields satisfy the equations [4],[5],

$$\begin{aligned} \nabla\times\mathbf{Q}_{L,\mathbf{r}_0} &= \gamma_L\mathbf{Q}_{L,\mathbf{r}_0} \\ \nabla\times\mathbf{Q}_{R,\mathbf{r}_0} &= -\gamma_R\mathbf{Q}_{L,\mathbf{r}_0} \end{aligned} \tag{2}$$

where γ_L and γ_R are wave numbers given by,

$$\gamma_L=\frac{k}{1-k\beta}\,,\ \gamma_R=\frac{k}{1+k\beta} \tag{3}$$

with $k=\omega(\varepsilon\mu)^{1/2}$, ω being the angular frequency. The indices L and R denote the LCP and RCP fields respectively. The spherical incident Beltrami fields with suitable normalization have the form [1], [2],

$$\mathbf{Q}^{inc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L) = \frac{1}{2}\,(\tilde{\mathbf{I}} + \frac{1}{\gamma_L^2}\nabla\nabla + \frac{1}{\gamma_L}\,\nabla\times\tilde{\mathbf{I}})\,(\frac{h_0\,(\gamma_L\left|\mathbf{r}-\mathbf{r}_0\right|)}{h_0\,(\gamma_L r_0)})\cdot\hat{\mathbf{p}}_L = \\ = \frac{2\pi r_0 \mathrm{e}^{-\mathrm{i}\gamma_L r_0}}{\gamma_L}\,[\gamma_L\tilde{\mathbf{G}}_{fs}(\mathbf{r},\mathbf{r}_0) + \nabla\times\tilde{\mathbf{G}}_{fs}(\mathbf{r},\mathbf{r}_0)\,]\cdot\hat{\mathbf{p}}_L \tag{4}$$

$$\mathbf{Q}^{inc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_R) = \frac{1}{2}\,(\tilde{\mathbf{I}} + \frac{1}{\gamma_R^2}\nabla\nabla - \frac{1}{\gamma_R}\,\nabla\times\tilde{\mathbf{I}})\,(\frac{h_0\,(\gamma_R\left|\mathbf{r}-\mathbf{r}_0\right|)}{h_0\,(\gamma_R r_0)})\cdot\hat{\mathbf{p}}_R = \\ = \frac{2\pi r_0 \mathrm{e}^{-\mathrm{i}\gamma_R r_0}}{\gamma_R}\,[\gamma_R\tilde{\mathbf{G}}_{fs}(\mathbf{r},\mathbf{r}_0) - \nabla\times\tilde{\mathbf{G}}_{fs}(\mathbf{r},\mathbf{r}_0)\,]\cdot\hat{\mathbf{p}}_R \tag{5}$$

where $h_0(x) = h_0^1(x) = \mathrm{e}^{\mathrm{i}x}/(\mathrm{i}x)$ is the zeroth-order spherical Hankel function of first kind, $\tilde{\mathbf{I}} = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\boldsymbol{\psi}}\hat{\boldsymbol{\psi}} + \hat{\mathbf{z}}\hat{\mathbf{z}}$ is the identity dyadic, $r_0 = \left|\mathbf{r}_0\right|$ and $\tilde{\mathbf{G}}_{fs}(\mathbf{r},\mathbf{r}_0)$ is the free space dyadic Green function [4]. The constant unit vectors $\hat{\mathbf{p}}_L$ and $\hat{\mathbf{p}}_R$ satisfy the relations

$$\hat{\mathbf{r}}_0\cdot\hat{\mathbf{p}}_L = \hat{\mathbf{r}}_0\cdot\hat{\mathbf{p}}_R = 0 \\ \hat{\mathbf{r}}_0\times\hat{\mathbf{p}}_L = \mathrm{i}\hat{\mathbf{p}}_L \\ \hat{\mathbf{r}}_0\times\hat{\mathbf{p}}_R = -\,\mathrm{i}\hat{\mathbf{p}}_R \tag{6}$$

We note that when $r_0\to\infty$, the incident electric field

$$\mathbf{E}^{inc}_{\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) = \\ = \mathbf{Q}^{inc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L) + \mathbf{Q}^{inc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_R) \tag{7}$$

reduces to plane electric wave with direction of propagation $-\hat{\mathbf{r}}_0$ and polarizations $\hat{\mathbf{p}}_L$, $\hat{\mathbf{p}}_R$, since

$$\lim_{r_0\to\infty}\mathbf{Q}^{inc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L) = e^{-\mathrm{i}\gamma_L\hat{\mathbf{r}}_0\cdot\mathbf{r}}\hat{\mathbf{p}}_L = \\ = \mathbf{Q}^{inc}_{L}(\mathbf{r};-\hat{\mathbf{r}}_0,\hat{\mathbf{p}}_L) \tag{8}$$

$$\lim_{r_0\to\infty}\mathbf{Q}^{inc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_R) = \mathrm{e}^{-\mathrm{i}\gamma_R\hat{\mathbf{r}}_0\cdot\mathbf{r}}\hat{\mathbf{p}}_R = \\ = \mathbf{Q}^{inc}_{R}(\mathbf{r};-\hat{\mathbf{r}}_0,\hat{\mathbf{p}}_R) \tag{9}$$

We consider $\mathbf{E}^{inc}_{\mathbf{r}_0}$ is incident upon a perfectly conducting sphere of radius a. Then, we want to calculate the scattered electric field $\mathbf{E}^{sc}_{\mathbf{r}_0}$, which is the unique solution of the following exterior boundary value problem

$$\nabla\times\nabla\times\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}) - 2\gamma^2\beta\nabla\times\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}) - \gamma^2\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}) = \mathbf{0},\ \ r > a \quad (10)$$

$$\hat{\mathbf{n}}\times\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}) = -\ \hat{\mathbf{n}}\times\mathbf{E}^{inc}_{\mathbf{r}_0}(\mathbf{r}),\ \ r = a \quad (11)$$

The Silver-Müller radiation condition is modified as follows

$$\hat{\mathbf{r}}\times\nabla\times\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}) - \beta\gamma^2\hat{\mathbf{r}}\times\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}) + \frac{\mathrm{i}\gamma^2}{k}\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}) = o(\frac{1}{r}),\ \ r\to\infty \quad (12)$$

uniformly in all directions $\hat{\mathbf{r}}\in S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$, $\hat{\mathbf{n}}$ is the outward normal unit vector on the scatterer and $\gamma^2 = \gamma_L\gamma_R$. The scattered electric field will be depended on the polarizations $\hat{\mathbf{p}}_L$, $\hat{\mathbf{p}}_R$ and will have the decomposition

$$\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) = \mathbf{Q}^{sc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) + \mathbf{Q}^{sc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) \quad (13)$$

where $\mathbf{Q}^{sc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ and $\mathbf{Q}^{sc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ are the corresponding scattered Beltrami fields which have the following behavior, when $r\to\infty$

$$\mathbf{Q}^{sc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) = h_0(\gamma_L r)\cdot \mathbf{g}_{L,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) + O(\frac{1}{r^2}) \quad (14)$$

$$\mathbf{Q}^{sc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) = h_0(\gamma_R r)\cdot \mathbf{g}_{R,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) + O(\frac{1}{r^2}) \quad (15)$$

The functions $\mathbf{g}_{L,\mathbf{r}_0}$ and $\mathbf{g}_{R,\mathbf{r}_0}$ are the LCP and RCP far-field patterns respectively [4], [6].

If either a LCP or a RCP spherical electric wave $\mathbf{E}^{inc}_{\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_A)$, $A = L, R$, is incident upon the scatterer, then the scattering cross–section, is given by [6],

$$\sigma^{sc}_{A,\mathbf{r}_0} = \int_{S^2}\Big[\frac{1}{\gamma_L^2}\left|\mathbf{g}_{L,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_A)\right|^2 + \frac{1}{\gamma_R^2}\left|\mathbf{g}_{R,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_A)\right|^2\Big]\,ds(\hat{\mathbf{r}}) \quad (16)$$

Exact Green’s function

We take spherical coordinates (r,θ,φ) where $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi)$, with the origin at the center of the spherical scatterer, so that the point source is at $r = r_0$, $\theta = 0$.

Thus, $\mathbf{r}_0 = r_0\hat{\mathbf{z}}$, $\hat{\mathbf{p}}_L = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - \mathrm{i}\hat{\boldsymbol{\psi}})$ and $\hat{\mathbf{p}}_R = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \mathrm{i}\hat{\boldsymbol{\psi}})$, where $\hat{\mathbf{x}}, \hat{\boldsymbol{\psi}}$ and $\hat{\mathbf{z}}$ are unit vectors in the x, ψ and z directions, respectively. Using spherical vector wave functions, [4], [5], [7], and taking into account (4), and (5), we obtain [2],

$$\mathbf{Q}^{inc}_{L,\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_L) = \sum_{n=1}^{\infty} B^L_n \{\mathbf{L}^{(1)}_{o1n}(\gamma_L \mathbf{r}) + + \mathrm{i}\mathbf{L}^{(1)}_{e1n}(\gamma_L \mathbf{r})\} \tag{17}$$

or

$$\mathbf{Q}^{inc}_{R,\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_R) = \sum_{n=1}^{\infty} B^R_n \{\mathbf{R}^{(1)}_{o1n}(\gamma_R \mathbf{r}) - - \mathrm{i}\mathbf{R}^{(1)}_{e1n}(\gamma_R \mathbf{r})\} \tag{18}$$

where, for $r < r_0$

$$B^A_n = \frac{1}{2\sqrt{2}h_0(\gamma_A r_0)} \frac{2n+1}{n(n+1)} H_n(\gamma_A r_0)$$

and (19)

$$H_n(\gamma_A r_0) = h_n(\gamma_A r_0) - \mathrm{i}\tilde{h}_n(\gamma_A r_0)$$

with $A = L, R$. The h_n is a spherical Hankel function of first order,

$$\tilde{h}(x) = x^{-1}h_n(x) + h'_n(x),$$

$\mathbf{L}^{(\rho)}_{s1n}$ and $\mathbf{R}^{(\rho)}_{s1n}$, with $s = e$, or o (even or odd) are the spherical functions [4], [5],

$$\begin{aligned} \mathbf{L}^{(\rho)}_{s1n}(\gamma_L \mathbf{r}) &= \mathbf{M}^{(\rho)}_{s1n}(\gamma_L \mathbf{r}) + \mathbf{N}^{(\rho)}_{s1n}(\gamma_L \mathbf{r}) \\ \mathbf{R}^{(\rho)}_{s1n}(\gamma_R \mathbf{r}) &= \mathbf{M}^{(\rho)}_{s1n}(\gamma_R \mathbf{r}) - \mathbf{N}^{(\rho)}_{s1n}(\gamma_R \mathbf{r}) \end{aligned} \tag{20}$$

where $\rho = 1, 3$, the $\mathbf{M}^{(\rho)}_{s1n}$ and $\mathbf{N}^{(\rho)}_{s1n}$ are known spherical vector function [7]. The scattered electric field that comes from a LCP incident field or a RCP incident field has a similar expansion to (17) or to (18)

$$\begin{aligned} \mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_L) &= \mathbf{Q}^{sc}_{L,\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_L) + \mathbf{Q}^{sc}_{R,\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_L) = \\ &= \sum_{n=1}^{\infty} B^L_n a^L_n \{\mathbf{L}^{(3)}_{o1n}(\gamma_L \mathbf{r}) + \mathrm{i}\mathbf{L}^{(3)}_{e1n}(\gamma_L \mathbf{r})\} + \sum_{n=1}^{\infty} B^L_n a^R_n \{\mathbf{R}^{(3)}_{o1n}(\gamma_R \mathbf{r}) + \mathrm{i}\mathbf{R}^{(3)}_{e1n}(\gamma_R \mathbf{r})\} \end{aligned} \tag{21}$$

or

$$\begin{aligned} \mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_R) &= \mathbf{Q}^{sc}_{L,\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_R) + \mathbf{Q}^{sc}_{R,\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_R) + \\ &= \sum_{n=1}^{\infty} B^R_n b^L_n \{\mathbf{L}^{(3)}_{o}(\gamma_L \mathbf{r}) - i\mathbf{L}^{(3)}_{e}(\gamma_L \mathbf{r})\} + \sum_{n=1}^{\infty} B^R_n b^R_n \{\mathbf{R}^{(3)}_{o}(\gamma_R \mathbf{r}) - \mathrm{i}\mathbf{R}^{(3)}_{e}(\gamma_R \mathbf{r})\} \end{aligned} \tag{22}$$

Using the boundary condition (11) on $r = a$, we obtain [2],

$$a_n^L = -\frac{j_n(\gamma_L a)\,\tilde{h}_n(\gamma_R a) + \tilde{j}_n(\gamma_L a)\,h_n(\gamma_R a)}{h_n(\gamma_L a)\,\tilde{h}_n(\gamma_R a) + \tilde{h}_n(\gamma_L a)\,h_n(\gamma_R a)} \tag{23}$$

and

$$a_n^R = -\frac{j_n(\gamma_L a)\,\tilde{h}_n(\gamma_L a) - \tilde{j}_n(\gamma_L a)\,h_n(\gamma_L a)}{h_n(\gamma_L a)\,\tilde{h}_n(\gamma_R a) + \tilde{h}_n(\gamma_L a)\,h_n(\gamma_R a)} \tag{24}$$

or

$$b_n^L = -\frac{j_n(\gamma_R a)\,\tilde{h}_n(\gamma_R a) - \tilde{j}_n(\gamma_R a)\,h_n(\gamma_R a)}{h_n(\gamma_L a)\,\tilde{h}_n(\gamma_R a) + \tilde{h}_n(\gamma_L a)\,h_n(\gamma_R a)} \tag{25}$$

and

$$b_n^R = -\frac{j_n(\gamma_R a)\,\tilde{h}_n(\gamma_L a) + \tilde{j}_n(\gamma_R a)\,h_n(\gamma_L a)}{h_n(\gamma_L a)\,\tilde{h}_n(\gamma_R a) + \tilde{h}_n(\gamma_L a)\,h_n(\gamma_R a)} \tag{26}$$

Using the asymptotic forms [4], [7],

$$\mathbf{L}_{s1n}^{(3)}(\gamma_L \mathbf{r}) \sim \sqrt{n(n+1)}(-\mathrm{i})^n\, h_0(\gamma_L r)\,\mathbf{f}_{s1n}^L(\hat{\mathbf{r}}) \tag{27}$$

$$\mathbf{R}_{s1n}^{(3)}(\gamma_R \mathbf{r}) \sim \sqrt{n(n+1)}(-\mathrm{i})^n\, h_0(\gamma_R r)\,\mathbf{f}_{s1n}^R(\hat{\mathbf{r}}) \tag{28}$$

where let us introduce LCP Beltrami angular $\mathbf{f}_{s1n}^L(\hat{\mathbf{r}})$, and RCP Beltrami angular $\mathbf{f}_{s1n}^R(\hat{\mathbf{r}})$ [4], satisfy by relations,

$$\mathbf{f}_{s1n}^L(\hat{\mathbf{r}}) = \mathbf{C}_{s1n}(\hat{\mathbf{r}}) + \mathrm{i}\mathbf{B}_{s1n}(\hat{\mathbf{r}})\,, \quad \mathbf{f}_{s1n}^R(\hat{\mathbf{r}}) = \mathbf{C}_{s1n}(\hat{\mathbf{r}}) - \mathrm{i}\mathbf{B}_{s1n}(\hat{\mathbf{r}}) \tag{29}$$

we calculate the electric far-field patterns [2],

$$\mathbf{g}_{A,\mathbf{r}_0}^{sc}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_L) = \sum_{n=1}^{\infty} \frac{(2n+1)(-\mathrm{i})^{n-1}}{2\sqrt{2n(n+1)}}\,\frac{H_n(\gamma_L r_0)}{h_0(\gamma_L r_0)}\,a_n^A\{\mathbf{f}_{e1n}^A(\hat{\mathbf{r}}) - \mathrm{i}\mathbf{f}_{o1n}^A(\hat{\mathbf{r}})\} \tag{30}$$

or

$$\mathbf{g}_{A,\mathbf{r}_0}^{sc}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_R) = \sum_{n=1}^{\infty} \frac{(2n+1)(-\mathrm{i})^{n-1}}{2\sqrt{2n(n+1)}}\,\frac{H_n(\gamma_R r_0)}{h_0(\gamma_R r_0)}\,b_n^A\{-\mathbf{f}_{e1n}^A(\hat{\mathbf{r}}) - \mathrm{i}\mathbf{f}_{o1n}^A(\hat{\mathbf{r}})\} \tag{31}$$

A far-field inverse problem

So far, all of our formulas are exact. In the asymptotic results to follow, there are three parameters $\gamma_A a$, with $A = L, R$ and $\tau = a/r_0$. We note that the geometrical parameter τ must satisfy $0 < \tau < 1$ because the point source is outside of the sphere.

We assume that $|\gamma_A a| \ll 1$, as well; that is we make the so-called low-frequency assumption. From (23), (24), (26), (27), we obtain [2],

$$a_n^L \sim \frac{1+\beta k}{2\mathrm{i}\zeta_n^2(2n+1)}(\gamma_L a)^{2n+1}$$

$$a_n^R \sim -\frac{\mathrm{i}}{2n\zeta_n^2}\frac{(1-\beta k)^{n+2}}{(1+\beta k)^{n+1}}(\gamma_L a)^{2n+1}$$

$$\gamma_L a \to 0 \tag{32}$$

or

$$b_n^L \sim -\frac{\mathrm{i}}{2n\zeta_n^2}\frac{(1+\beta k)^{n+2}}{(1-\beta k)^{n+1}}(\gamma_R a)^{2n+1}$$

$$b_n^R \sim -\frac{\mathrm{i}(1-\beta k)}{2\zeta_n^2(2n+1)}(\gamma_R a)^{2n+1}$$

$$\gamma_R a \to 0 \tag{33}$$

where

$\zeta_n = 1\cdot 3\cdot 5\cdots(2n-1) = (2n)!/(2^n n!)$.
In particular,

$$\begin{cases} a_1^L = -\mathrm{i}\,\dfrac{1+\beta k}{6}(\gamma_L a)^3 + O((\gamma_L a)^5) \\ a_2^L = -\mathrm{i}\,\dfrac{1+\beta k}{90}(\gamma_L a)^5 + O((\gamma_L a)^7) \end{cases}$$

$$\gamma_L a \to 0 \tag{34}$$

$$\begin{cases} a_1^R = -\dfrac{\mathrm{i}(1-\beta k)^3}{2(1+\beta k)^2}(\gamma_L a)^3 + O((\gamma_L a)^5) \\ a_2^R = -\dfrac{\mathrm{i}(1-\beta k)^4}{36(1+\beta k)^3}(\gamma_L a)^5 + O((\gamma_L a)^7) \end{cases}$$

$$\gamma_L a \to 0 \tag{35}$$

or

$$\begin{cases} b_1^L = -\dfrac{\mathrm{i}(1+\beta k)^3}{2(1-\beta k)^2}(\gamma_R a)^3 + O((\gamma_L a)^5) \\ b_2^L = -\dfrac{\mathrm{i}(1+\beta k)^4}{36(1-\beta k)^3}(\gamma_R a)^5 + O((\gamma_L a)^7) \end{cases}$$

$$\gamma_R a \to 0 \tag{36}$$

$$\begin{cases} b_1^R = -\dfrac{\mathrm{i}(1-\beta k)}{6}(\gamma_R a)^3 + O((\gamma_R a)^5) \\ b_2^R = -\dfrac{\mathrm{i}(1-\beta k)}{90}(\gamma_R a)^5 + O((\gamma_R a)^7) \end{cases}$$

$$\gamma_R a \to 0 \tag{37}$$

In order to calculate $\mathbf{g}^{sc}_{A,\mathbf{r}_0}$ with an error of $O((\gamma_A a)^4)$ we only need the following [4]

$$\mathbf{C}_{o11}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{2}}\,(\cos\varphi\,\hat{\boldsymbol{\theta}} - \cos\theta\,\sin\varphi\,\hat{\boldsymbol{\varphi}})$$

$$\mathbf{B}_{o11}(\hat{\mathbf{r}}) = -\frac{1}{\sqrt{2}}\,(\cos\theta\,\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}) \qquad (38)$$

$$\mathbf{B}_{e11}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{2}}\,(\cos\theta\,\sin\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}})$$

$$\mathbf{C}_{e11}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{2}}\,(-\sin\varphi\,\hat{\boldsymbol{\theta}} - \cos\theta\,\cos\varphi\,\hat{\boldsymbol{\varphi}}) \qquad (39)$$

$$\mathbf{C}_{o12}(\hat{\mathbf{r}}) = (3/2)^{1/2}\cdot(\cos\theta\,\cos\varphi\,\hat{\boldsymbol{\theta}} - \cos 2\theta\,\sin\varphi\,\hat{\boldsymbol{\varphi}}) \qquad (40)$$

$$\mathbf{B}_{e12}(\hat{\mathbf{r}}) = (3/2)^{1/2}\cdot(\cos 2\theta\,\cos\varphi\,\hat{\boldsymbol{\theta}} - \cos\theta\,\sin\varphi\,\hat{\boldsymbol{\varphi}}) \qquad (41)$$

$$\mathbf{B}_{o12}(\hat{\mathbf{r}}) = (3/2)^{1/2}\cdot(\cos 2\theta\,\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\theta\,\cos\varphi\,\hat{\boldsymbol{\varphi}}) \qquad (42)$$

$$\hat{\mathbf{C}}_{e12}(\hat{\mathbf{r}}) = (3/2)^{1/2}\cdot(-\cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} - \cos 2\theta\cos\varphi\,\hat{\boldsymbol{\varphi}}) \qquad (43)$$

So from (30) and (31), we finally obtain [2],

$$\begin{aligned}
\mathbf{g}^{sc}_{A,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_A) = \varpi_A\,\frac{(1-\varpi_A\beta k)}{8}\,[-(\gamma_A a)\tau^2[\mathbf{f}^A_{e11}(\hat{\mathbf{r}}) + \varpi_A \mathrm{i}\mathbf{f}^A_{o11}(\hat{\mathbf{r}})] + \\
+(\gamma_A a)^2\{2\mathrm{i}\tau[\mathbf{f}^A_{e11}(\hat{\mathbf{r}}) + \varpi_A \mathrm{i}\mathbf{f}^A_{o11}(\hat{\mathbf{r}})] + \frac{\mathrm{i}}{\sqrt{3}}\,\tau^3[\mathbf{f}^A_{e12}(\hat{\mathbf{r}}) + \varpi_A \mathrm{i}\mathbf{f}^A_{o12}(\hat{\mathbf{r}})]\} + \\
+(\gamma_A a)^3\{2[\mathbf{f}^A_{e11}(\hat{\mathbf{r}}) + \varpi_A \mathrm{i}\mathbf{f}^A_{o11}(\hat{\mathbf{r}})] + \frac{4}{3\sqrt{3}}\,\tau^2[\mathbf{f}^A_{e12}(\hat{\mathbf{r}}) + \varpi_A \mathrm{i}\mathbf{f}^A_{o12}(\hat{\mathbf{r}})]\}] + \\
+O((\gamma_A a)^4)
\end{aligned} \qquad (44)$$

and

$$\begin{aligned}
\mathbf{g}^{sc}_{A,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_{A^c}) = \varpi_A\,\frac{3(1-\varpi_A\beta k)^2}{8(1+\varpi_A\beta k)}\,(\gamma_A a)\tau^2[\mathbf{f}^A_{e11}(\hat{\mathbf{r}}) - \varpi_A \mathrm{i}\mathbf{f}^A_{o11}(\hat{\mathbf{r}})] + \\
+(\gamma_A a)^2\{-\varpi_A\,\frac{3\mathrm{i}(1-\varpi_A\beta k)}{4}\,\tau[\mathbf{f}^A_{e11}(\hat{\mathbf{r}}) - \varpi_A \mathrm{i}\mathbf{f}^A_{o11}(\hat{\mathbf{r}})] - \\
-\varpi_A\,\frac{5\mathrm{i}}{16\sqrt{3}}\,\frac{(1-\varpi_A\beta k)^2}{1+\varpi_A\beta k}\,\tau^3[\mathbf{f}^A_{e12}(\hat{\mathbf{r}}) - \varpi_A \mathrm{i}\mathbf{f}^A_{o12}(\hat{\mathbf{r}})]\} + \\
+(\gamma_A a)^3\{-\varpi_A\,\frac{3(1+\varpi_A\beta k)}{4}\,[\mathbf{f}^A_{e11}(\hat{\mathbf{r}}) - \varpi_A \mathrm{i}\mathbf{f}^A_{o11}(\hat{\mathbf{r}})] -
\end{aligned} \qquad (45)$$

$$- \varpi_A \frac{5(1-\varpi_A \beta k)}{12\sqrt{3}} \tau^2 [\mathbf{f}^A_{e12}(\hat{\mathbf{r}}) - \varpi_A \mathrm{i} \mathbf{f}^A_{o12}(\hat{\mathbf{r}})]\} + O((\gamma_A a)^4)$$

where $\varpi_A = \begin{cases} -1, & A = L \\ 1, & A = R \end{cases}$ and if $A = L, R$ at that case $A^c = R, L$.

Now the scattering cross-section, by LCP or RCP spherical Beltrami fields, from (16), is given by the relations [2],

$$\sigma^{sc}_{L,\mathbf{r}_0} = \int_{S^2} [\frac{1}{\gamma_L^2} \left|\mathbf{g}^{sc}_{L,\mathbf{r}_0}(\hat{\mathbf{r}} \mid \hat{\mathbf{p}}_L)\right|^2 + \frac{1}{\gamma_R^2} \left|\mathbf{g}^{sc}_{R,\mathbf{r}_0}(\hat{\mathbf{r}} \mid \hat{\mathbf{p}}_L)\right|^2] \, ds(\hat{\mathbf{r}}) =$$

$$= (\pi a^2)\{\frac{(1+\beta k)^2}{64} [\frac{16}{3} \tau^4 + (\gamma_L a)^2 (\frac{64}{3} \tau^2 + 16\tau^6) + (\gamma_L a)^4 (\frac{64}{3} + \frac{16}{9} \tau^4)] +$$

$$+ \frac{(1-\beta k)^4}{(1+\beta k)^2} [\frac{3}{4} \tau^4 + (\gamma_L a)^2 (3\tau^2 + \frac{5(1-\beta k)^2}{16(1+\beta k)^2} \tau^6) +$$

$$+ (\gamma_L a)^4 (3 + \frac{5(1-\beta k)}{9(1+\beta k)^2} \tau^4)]\} + O((\gamma_L a)^6), \ \gamma_L a \to 0 \qquad (46)$$

or

$$\sigma^{sc}_{R,\mathbf{r}_0} = \int_{S^2} [\frac{1}{\gamma_L^2} \left|\mathbf{g}^{sc}_{L,\mathbf{r}_0}(\hat{\mathbf{r}} \mid \hat{\mathbf{p}}_R)\right|^2 + \frac{1}{\gamma_R^2} \left|\mathbf{g}^{sc}_{R,\mathbf{r}_0}(\hat{\mathbf{r}} \mid \hat{\mathbf{p}}_R)\right|^2] \, ds(\hat{\mathbf{r}}) =$$

$$= (\pi a^2)\{\frac{(1-\beta k)^2}{64} [\frac{16}{3} \tau^4 + (\gamma_R a)^2 (\frac{64}{3} \tau^2 + 16\tau^6) + (\gamma_R a)^4 (\frac{64}{3} + \frac{16}{9} \tau^4)] +$$

$$+ \frac{(1+\beta k)^4}{(1-\beta k)^2} [\frac{3}{4} \tau^4 + (\gamma_R a)^2 (3\tau^2 + \frac{5(1+\beta k)^2}{16(1-\beta k)^2} \tau^6) +$$

$$+ (\gamma_R a)^4 (3 + \frac{5(1+\beta k)}{9(1-\beta k)^2} \tau^4)]\} + O((\gamma_R a)^6), \ \gamma_R a \to 0 \qquad (47)$$

In the special case $r_0 \to \infty$ $(\tau \to 0)$, by the relations (46), (47) we obtain [2],

$$\sigma = (\pi a^2)\{\frac{(1+\beta k)^2}{3} + \frac{3(1-\beta k)^4}{(1+\beta k)^2}\} (\gamma_L a)^4 + O((\gamma_L a)^6), \ \gamma_L a \to 0 \qquad (48)$$

or

$$\sigma = (\pi a^2)\{\frac{(1-\beta k)^2}{3} + \frac{3(1+\beta k)^4}{(1-\beta k)^2}\}(\gamma_R a)^4 + O((\gamma_R a)^6),\ \gamma_R a \to 0 \quad (49)$$

likewise in the case $\gamma_L a \to 0$ and $\gamma_R a \to 0$, by the relations (46), (47) we obtain [2],

$$\sigma^{sc}_{A,\mathbf{r}_0} = \frac{1}{4} f_A(\beta, k)\,(\pi a^2)\,(a/r_0)^4 \quad (50)$$

where $A = L, R$, with $f_A(\beta, k) = \dfrac{(1-\varpi_A \beta k)^2}{3} + \dfrac{3(1+\varpi_A \beta k)^4}{(1-\varpi_A \beta k)^2}$ (51)

Choose a Cartesian coordinate system Oxψz, and five point-source locations, namely (0,0,0), (l,0,0), (0,l,0) (0,0,l) and (0,0,2l), which are at (unknown) distances r_0, r_1, r_2, r_3 and r_4, respectively from the sphere's center. The parameter l is a chosen fixed length. For each location, measure the leading order term in the low-frequency expansion of the scattering cross-section.

Thus, our five measurements are [2],

$$m_j = \frac{1}{4} f_A(\beta, k)\pi a^2 (\frac{a}{r_j})^4 \quad (52)$$

$$j = 0, 1, 2, 3, 4$$

Dimensionless quantities related to m_j are

$$\gamma_j = \frac{l}{\sqrt{m_j}} = 2\sqrt{\frac{1}{f_A(\beta, k)\pi}}\,\frac{l}{a}\,(\frac{r_j}{a})^2 \quad (53)$$

$$j = 0, 1, 2, 3, 4$$

equivalently, we obtain [2],

$$r_j^2 = \frac{1}{2}\sqrt{f_A(\beta, k)\pi}\,\frac{a^3}{l}\,\gamma_j \quad (54)$$

$$j = 0, 1, 2, 3, 4$$

There are six unknowns namely r_0, r_1, r_2, r_3, r_4 and a. Furthermore, r_0, r_3 and r_4 are related using the cosine rule, $r_4^2 + r_0^2 = 2r_3^2 + 2l^2$. So, we can find the six unknowns. The center of the spherical scatterer is obtained from the intersection of the four spheres centers at (0,0,0) (l,0,0), (0,l,0) and (0,0,l), with corresponding radius r_0, r_1, r_2, r_3 respectively.

References

[1] Athanasiadis C., Berketis N., "Scattering relations for point – source excitation in chiral media", Mathematical Methods in the Applied Sciences, Vol. 29, 2006,pp. 27-48
[2] Berketis M. Nikolaos, "Scattering spherical Electro-magnetic waves in chiral media", PhD thesis, University of Athens – Department of Mathematics, December 2007
[3] Arnaoudov Y, Dassios G, Hadjinicolaou M., "The resistive coated sphere in the presence of a point generated wave field", Mathematical Methods in the Applied Sciences, Vol. 22, 1999, pp. 73-90
[4] Lakhtakia A., "Beltrami Fields in Chiral Media", World Scientific, Singapore, 1994: §2-1.3 (pp.42-44), §2-2.3 (pp.49-52), §4-1 (p.130 Chapter 4), §5-1 (p.164 Chapter 5), §6-1.1 (pp.214-397), §8-2.1 (pp.391-397), §9-2.3 (pp.468- 473).
[5] Lakhtakia A, Varadan VK, Varadan VV., "Time – Harmonic Electromagnetic Fields in Chiral Media", Lecture Notes in Physics, Vol. 335, Springer, Berlin, 1989: §3-§4-§7-§11-§12-§13
[6] Athanasiadis C., "On the far field patterns for electro-magnetic scattering by a chiral obstacle in a chiral en-vironment", Journal of Mathematical Analysis and Applications, Vol. 309, 2005, pp. 517-533
[7] Morse P.M, Feshbach H., "Methods of Theoretical Physics, Part II", Mc Graw-Hill, New York, 1953: pp. 1864-1866, 1870, 1871, 1887

Preprint Versions

Direct and inverse scattering problems for spherical
Electromagnetic waves in chiral media
Christodoulos Athanasiadis and Nikolaos Berketis
arxiv.org/abs/0812.2169

* About the Author

Nikolaos Berketis, was born in Athens, Greece, 1952. Mathematics degree, Aristotle University of Thessaloniki, 1976. MSc in Applied Mathematics-Numerical Functional Analysis, Department of Mathematics, National and Kapodistrian University of Athens (UoA), 2003. Ph.D. in Applied Mathematics, Department of Mathematics, UoA, 2007. Professor of Secondary Education, 1980-2006. High school Principal, 2007-2012. Teacher at the UoA: Department of Informatics and Telecommunications, 2002-2003. Postgraduate program of the Department of Geology and Paleontology, 2003-2004. Research experience: Participation in the following research projects: 1) "The far-field operator in solving inverse scattering problem". Special Research Account 70/4/6412, UoA, 2002. 2) "Scattering of spherical electromagnetic waves by chiral materials". Special Research Account 70/4/6412, UoA, 2004. 3) "Mathematical analysis of wave propagation in chiral electromagnetic and elastic media." Pythagoras II: Reinforcing of University research groups (EPEAEK II), Department of Mathematics, UoA, 2005. Member of the Question Bank department of Evaluation and Training Research Center, Hellenic Mathematical Society (HMS), 2001-2004. Editorial Board experience: Member of the Scientific Committee of Euclid C΄ journal, Hellenic Mathematical Society (HMS). Member of the Scientific Committee of the project "Digital Training Textbooks for Pan-Hellenic examinations" -for Mathematics, Science and Technology Educational Path, 3rd grade, High School and Mathematics & Elementary Statistics, General Education, 3rd grade, High School- implemented by Computer Technology Institute University of Patra, for the Ministry of Education. Eighteen published peer-reviewed papers, more than 150 references, in peer-reviewed journals, utilizing both Hellenic and international bibliography. Participation in the project: "Change of speed of pulse wave after surgical narrowing of the descending thoracic aorta, "1st prize, Best Project award, 28th Annual Pan-Hellenic Medical Conference in the field of basic research, oral presentation, Athens, May 21-25, 2002. Personal work on the scattering of spherical electromagnetic waves in chiral media is posted at "The Smithsonian/NASA Astrophysics Data System" website. Authoring experience: Scientific research articles and books.

nberketis@gmail.com