Spherical Beltrami Fields in Chiral Media: Reciprocity and General Theorems

Nikolaos M. Berketis *

Independent Researcher, Athens, Greece

Abstract

The Beltrami fields are extremely important for the description of time-harmonic electromagnetic fields in chiral media. This paper introduces the spherical Beltrami fields in chiral media, extending the corresponding known results for plane electromagnetic waves. Two key theorems are given for the scattering of spherical electromagnetic waves in chiral environment from a perfect conductor: the Reciprocity and the General Scattering theorems.

Keywords

Chiral media, spherical Beltrami fields, reciprocity relation, general scattering theorem

Introduction

The name "Beltrami field" comes from the fact that these fields **A** satisfy the equation type $\nabla \times \mathbf{A} = \alpha \mathbf{A}$, where $\alpha \neq 0$ is a reciprocal length [1]. The word chiral regards the concept of "chirality", derived from the Greek word hand. In geometry, a figure is chiral if it cannot be mapped to its mirror image by rotations and translations alone. Typical examples are human hands, snail shells, spirals and spirals in general.

The discoveries of natural optical activity in special materials have been known since the beginning of last century. Though optical activity has been considered in optics and in quantum mechanics for many years, its analysis within the framework of classical electromagnetic field theory arose much later. Recently, there has been a considerable interest in the study of scattering and diffraction by chiral medium. Chiral media are isotropic birefringent substances that respond to either electric or magnetic excitation with both electric and magnetic polarizations. Such media have been known since the end of the nineteenth century (e.g. the study of chirality by Pasteur) and find a wide range of applications from many sciences [1].

In general, the electromagnetic fields inside the chiral medium are governed by Maxwell equations together with Drude-Born-Fedorov equations in which the electric and magnetic fields are coupled [1]. The chiral medium is characterized by the electric permittivity ε, magnetic permeability μ and chirality measure β.

Scattering problems with incident waves have been studied: for spherical acoustic waves in the works [2]-[4], as well in the works [2], [5] for spherical electromagnetic waves in non-chiral electromagnetic waves and finally for spherical electromagnetic waves in chiral media in the works [6], [7].

In this work, we derive reciprocity relation and general scattering theorem, from spherical electromagnetic waves emanating from point sources in chiral media and scattered by a perfect conductor. Similar theorems are proved, analogously, for a chiral dielectric [6], [7].

As it is well known [1], in a homogeneous isotropic chiral medium the electromagnetic field is composed of left-circularly polarized (LCP) and right-circularly polarized (RCP) components which are propagated independently and with different phase speeds. When either a LCP or a RCP or a linear combination of LCP and RCP electromagnetic waves is incident upon a chiral scatterer then the scattered field is composed of both LCP and RCP components. So, using Bohren decomposition [1], [6], [7], the electromagnetic waves are expressed in terms of LCP and RCP Beltrami fields. The use of the Beltrami fields for problems of scattering in chiral material leads to a more simplified relationship.

This is due to the fact that the differential equations which satisfy Beltrami fields is of first order while the electric and magnetic field satisfy the Helmholtz modified equation which is of second-order. Moreover, the Beltrami equations and the conditions of radiation for the Beltrami fields are in effect separately for each one of them but on the scatterer surface, when applying the boundary conditions, both Beltrami fields are present. This fact makes easier to use these equations mainly in scattering problems where the behavior of the scattered field away from the scatterer is studied.

In the second section, we formulate the problem for spherical electromagnetic wave in a chiral medium. In the third section we formulate the problem of electromagnetic wave scattering from a perfect conductor as a function of the Beltrami fields using the Bohren transformation. The spherical Beltrami fields are defined in the fourth section. Finally, Reciprocity relation and the General Scattering theorem are given in fifth section.

Problem Formulation

Let Ω^- be a bounded and closed subset of $\mathbb{R}^3$ having a C^2-boundary S, i.e. $\partial\Omega^- = S$. The set Ω^- will be referred to as the scatterer. The exterior $\Omega^+ = \mathbb{R}^3/\Omega^-$ of the scatterer is an infinite isotropic homogeneous chiral medium with electric permittivity ε, magnetic permeability μ and chirality measure β.

The scatterer is filled with a isotropic chiral medium with corresponding physical parameters ε^-, μ^- and β^-. All the physical parameters are assumed to be real positive constants.

Let $(\mathbf{E}^{inc}_{\mathbf{r}_0}, \mathbf{H}^{inc}_{\mathbf{r}_0})$ be a time-Harmonic spherical electromagnetic wave, due a point source located at a point with position vector $\mathbf{r}_0$ with respect to an origin O in the vicinity of Ω^-.

This wave is incident upon the scatterer Ω^- and let $(\mathbf{E}^{sc}_{\mathbf{r}_0}, \mathbf{H}^{sc}_{\mathbf{r}_0})$ be the corresponding scattered field. Then the total electromagnetic field $(\mathbf{E}^{t}_{\mathbf{r}_0}, \mathbf{H}^{t}_{\mathbf{r}_0})$ in Ω^+ is given by

$$\begin{cases} \mathbf{E}^{t}_{r_0}(\mathbf{r}) = \mathbf{E}^{inc}_{r_0}(\mathbf{r}) + \mathbf{E}^{sc}_{r_0}(\mathbf{r}) \\ \mathbf{H}^{t}_{r_0}(\mathbf{r}) = \mathbf{H}^{inc}_{r_0}(\mathbf{r}) + \mathbf{H}^{sc}_{r_0}(\mathbf{r}) \end{cases} \quad (1)$$

We assume that the scattered field satisfies the Silver-Müller radiation condition

$$\frac{1}{\eta}\mathbf{E}^{sc}_{r_0}(\mathbf{r}) + \hat{\mathbf{r}} \times \mathbf{H}^{sc}_{r_0}(\mathbf{r}) = o(r^{-1}), \quad r \to \infty \quad (2)$$

uniformly in all directions, $\hat{\mathbf{r}} \in S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$, $r = |\boldsymbol{r}|$, $\hat{\mathbf{r}} = \boldsymbol{r}/r$ and $\eta = (\mu/\varepsilon)^{1/2}$ is the intrinsic impedance of the chiral medium in Ω^+. It is known [8], that (2) can be replaced by

$$\hat{\mathbf{r}} \times \mathbf{E}^{sc}_{r_0}(\mathbf{r}) - \eta\mathbf{H}^{sc}_{r_0}(\mathbf{r}) = o(r^{-1}), \quad r \to \infty \quad (3)$$

uniformly in all directions, $\hat{\mathbf{r}} \in S^2$. The total electric field $\mathbf{E}^{tot}$ satisfies the boundary condition:

$$\hat{\mathbf{n}} \times \mathbf{E}^{t}(\mathbf{r}) = \mathbf{0}, \ \mathbf{r} \in S \quad (4)$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the outer surface S. In view of the Drude-Born-Fedorov constitutive relations [1], the total exterior electromagnetic field satisfies in the source-free region Ω^+ the modified Maxwell equations

$$\nabla\times\mathbf{E}^t_{\mathbf{r}_0}(\mathbf{r}) = \beta\gamma^2\mathbf{E}^t_{\mathbf{r}_0}(\mathbf{r}) + + i\omega\mu(\frac{\gamma}{k})^2\mathbf{H}^t_{\mathbf{r}_0}(\mathbf{r}) \tag{5}$$

$$\nabla\times\mathbf{H}^t_{\mathbf{r}_0}(\mathbf{r}) = \beta\gamma^2\mathbf{H}^t_{\mathbf{r}_0}(\mathbf{r}) + - i\omega\varepsilon(\frac{\gamma}{k})^2\mathbf{E}^t_{\mathbf{r}_0}(\mathbf{r}) \tag{6}$$

where

$$k^2 = \omega^2\varepsilon\mu\,,\quad \gamma^2 = \frac{k^2}{1-k^2\beta^2} \tag{7}$$

with $|\beta k| < 1$, [1].

We note that (in contrast to the non-chiral case) k is not a wave number, but just a parameter without physical significance which has dimensions of inverse length and $\omega > 0$ is the angular frequency. The system of relations (5), (6) can be written as

$$\begin{bmatrix}\nabla\times\mathbf{E}^t_{\mathbf{r}_0}\\ \nabla\times\mathbf{H}^t_{\mathbf{r}_0}\end{bmatrix} = (\frac{\gamma}{k})^2\begin{bmatrix}k^2\beta & i\omega\mu\\ -i\omega\varepsilon & k^2\beta\end{bmatrix}\begin{bmatrix}\mathbf{E}^t_{\mathbf{r}_0}\\ \mathbf{H}^t_{\mathbf{r}_0}\end{bmatrix} \tag{8}$$

Beltrami Fields

As it is well known [1], [10], in chiral media LCP and RCP waves can both propagate independently and with different phase speeds. So, we consider the Bohren [9], decomposition of $\mathbf{E}_{\mathbf{r}_0}$ and $\mathbf{H}_{\mathbf{r}_0}$ into suitable Beltrami fields $\mathbf{Q}_{L,\mathbf{r}_0}$ and $\mathbf{Q}_{R,\mathbf{r}_0}$, as follows

$$\mathbf{E}^t_{\mathbf{r}_0} = \mathbf{Q}_{L,\mathbf{r}_0} + \mathbf{Q}_{R,\mathbf{r}_0} \tag{9}$$

$$\mathbf{H}^t_{\mathbf{r}_0} = \frac{1}{i\eta}(\mathbf{Q}_{L,\mathbf{r}_0} - \mathbf{Q}_{R,\mathbf{r}_0}) \tag{10}$$

where $\eta = (\mu/\varepsilon)^{1/2}$ is the impedance of the material.

Because the transformation Bohren the relation (8) is written:

$$\begin{bmatrix}\nabla\times\mathbf{Q}_{L,\mathbf{r}_0}\\ \nabla\times\mathbf{Q}_{R,\mathbf{r}_0}\end{bmatrix} = \begin{bmatrix}\gamma_L & 0\\ 0 & -\gamma_R\end{bmatrix}\begin{bmatrix}\mathbf{Q}_{L,\mathbf{r}_0}\\ \mathbf{Q}_{R,\mathbf{r}_0}\end{bmatrix} \tag{11}$$

or equivalent

$$\begin{cases}\nabla\times\mathbf{Q}_{L,\mathbf{r}_0} = \gamma_L\mathbf{Q}_{L,\mathbf{r}_0}\\ \nabla\times\mathbf{Q}_{R,\mathbf{r}_0} = -\gamma_R\mathbf{Q}_{R,\mathbf{r}_0}\end{cases} \tag{12}$$

where γ_L, γ_R are wave numbers for Beltrami fields $\mathbf{Q}_{L,\mathbf{r}_0}, \mathbf{Q}_{R,\mathbf{r}_0}$ and are given by,

$$\gamma_L = \frac{k}{1-k\beta}\,,\ \gamma_R = \frac{k}{1+k\beta} \tag{13}$$

respectively. Applying the transformation Bohren we split the field $\mathbf{E}^t_{\mathbf{r}_0}$, $\mathbf{H}^t_{\mathbf{r}_0}$ into the Beltrami fields $\mathbf{Q}_L$, $\mathbf{Q}_R$, so that the problem (2)-(6) becomes equivalent to the following problem: We are looking for $\mathbf{Q}_L$, $\mathbf{Q}_R$ in Ω^+ such that

$$\nabla \times \mathbf{Q}_L(\mathbf{r}) = \gamma_L \mathbf{Q}_L(\mathbf{r}), \ \mathbf{r} \in \Omega^+ \qquad (14)$$

$$\nabla \times \mathbf{Q}_R(\mathbf{r}) = -\gamma_R \mathbf{Q}_R(\mathbf{r}), \ \mathbf{r} \in \Omega^+ \qquad (15)$$

$$\hat{\mathbf{n}} \times \mathbf{Q}_L(\mathbf{r}) = \mathrm{i}\sqrt{\frac{\mu}{\varepsilon}}\ \hat{\mathbf{n}} \times \mathbf{Q}_R(\mathbf{r}), \ \mathbf{r} \in S \qquad (16)$$

$$\hat{\mathbf{r}} \times \mathbf{Q}_L(\mathbf{r}) + \mathrm{i}\mathbf{Q}_L(\mathbf{r}) = o(\frac{1}{r}), \ r \to \infty \qquad (17)$$

$$\hat{\mathbf{r}} \times \mathbf{Q}_R(\mathbf{r}) - \mathrm{i}\mathbf{Q}_R(\mathbf{r}) = o(\frac{1}{r}), \ r \to \infty \qquad (18)$$

Spherical Beltrami Fields

So, we consider the Bohren decomposition of $\mathbf{E}^{inc}_{\mathbf{r}_0}$ and $\mathbf{H}^{inc}_{\mathbf{r}_0}$ into suitable incident spherical Beltrami fields $\mathbf{Q}^{inc}_{L,\mathbf{r}_0}$ and $\mathbf{Q}^{inc}_{R,\mathbf{r}_0}$ which have the form [6], [7],

$$\mathbf{Q}^{inc}_{L,\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_L) = A_L \tilde{\mathbf{B}}_L(\mathbf{r}, \mathbf{r}_0)\cdot\hat{\mathbf{p}}_L \qquad (19)$$

$$\mathbf{Q}^{inc}_{R,\mathbf{r}_0}(\mathbf{r} \mid \hat{\mathbf{p}}_R) = A_R \tilde{\mathbf{B}}_R(\mathbf{r}, \mathbf{r}_0)\cdot\hat{\mathbf{p}}_R \qquad (20)$$

where

$$\tilde{\mathbf{B}}_L(\mathbf{r}, \mathbf{r}_0) = \frac{\mathrm{i}k\gamma_L}{8\pi\gamma^2}(\gamma_L \tilde{\mathbf{I}} + \frac{1}{\gamma_L}\nabla\nabla + \nabla \times \tilde{\mathbf{I}})h(\gamma_L\left|\mathbf{r} - \mathbf{r}_0\right|) \qquad (21)$$

$$\tilde{\mathbf{B}}_R(\mathbf{r}, \mathbf{r}_0) = \frac{\mathrm{i}k\gamma_R}{8\pi\gamma^2}(\gamma_R \tilde{\mathbf{I}} + \frac{1}{\gamma_R}\nabla\nabla - \nabla \times \tilde{\mathbf{I}})h(\gamma_R\left|\mathbf{r} - \mathbf{r}_0\right|) \qquad (22)$$

and $h(x) = \mathrm{e}^{\mathrm{i}x}/(\mathrm{i}x)$ is the zeroth order spherical Hankel function of the first kind and $\tilde{\mathbf{I}} = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\boldsymbol{\psi}}\hat{\boldsymbol{\psi}} + \hat{\mathbf{z}}\hat{\mathbf{z}}$ is the identity dyadic. We recall that $\tilde{\mathbf{B}}_L$ and $\tilde{\mathbf{B}}_R$ are the fundamental Green dyadics, for the Beltrami fields [1], [11]. The constant unit vectors $\hat{\mathbf{p}}_L$ and $\hat{\mathbf{p}}_R$ are assumed to satisfy the relations:

$$\hat{\mathbf{r}}_0 \cdot \hat{\mathbf{p}}_L = \hat{\mathbf{r}}_0 \cdot \hat{\mathbf{p}}_R = 0$$
$$\hat{\mathbf{r}}_0 \times \hat{\mathbf{p}}_L = \mathrm{i}\hat{\mathbf{p}}_L \qquad (23)$$
$$\hat{\mathbf{r}}_0 \times \hat{\mathbf{p}}_R = -\mathrm{i}\hat{\mathbf{p}}_R$$

The constants A_L and A_R are evaluated so that as the location of the point source goes to infinity along the ray in the direction $\hat{\mathbf{r}}_0$, the spherical fields degenerate into plane LCP and RCP Beltrami fields propagated in a direction from $\mathbf{r}_0$ towards O, with polarizations $\hat{\mathbf{p}}_L$ and $\hat{\mathbf{p}}_R$, respectively. Using the asymptotic forms:

$$\left|\mathbf{r} - \mathbf{r}_0\right| = r_0 - \hat{\mathbf{r}}_0\cdot\mathbf{r} + O(\frac{1}{r_0}),$$
$$\frac{\mathbf{r} - \mathbf{r}_0}{\left|\mathbf{r} - \mathbf{r}_0\right|} = -\hat{\mathbf{r}}_0 + O(\frac{1}{r_0}), \qquad (24)$$
$$r_0 \to \infty$$

in the relations (19) and (20) we obtain [6], [7],

$$\mathbf{Q}^{inc}_{j,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_j) = A_j \frac{2e^{-\mathrm{i}\gamma_j r_0}}{r_0}\tilde{\mathbf{K}}_j(-\hat{\mathbf{r}}_0)\cdot \cdot\hat{\mathbf{p}}_j e^{-\mathrm{i}\gamma_j \hat{\mathbf{r}}_0\cdot\mathbf{r}} + O(\frac{1}{r_0^2}), \quad r_0\to\infty \tag{25}$$

for $j = L, R$, where

$$\tilde{\mathbf{K}}_L(-\hat{\mathbf{r}}_0) = \frac{1}{2}(\tilde{\mathbf{I}} - \hat{\mathbf{r}}_0\hat{\mathbf{r}}_0 - \mathrm{i}\hat{\mathbf{r}}_0\times\tilde{\mathbf{I}}) \tag{26}$$

$$\tilde{\mathbf{K}}_R(-\hat{\mathbf{r}}_0) = \frac{1}{2}(\tilde{\mathbf{I}} - \hat{\mathbf{r}}_0\hat{\mathbf{r}}_0 + \mathrm{i}\hat{\mathbf{r}}_0\times\tilde{\mathbf{I}}) \tag{27}$$

In view of (23) the dyadics $\tilde{\mathbf{K}}_L$ and $\tilde{\mathbf{K}}_R$ satisfy the relations

$$\begin{cases}\hat{\mathbf{p}}_L = \tilde{\mathbf{K}}_L(-\hat{\mathbf{r}}_0)\cdot\hat{\mathbf{p}}_L\\ \hat{\mathbf{p}}_R = \tilde{\mathbf{K}}_R(-\hat{\mathbf{r}}_0)\cdot\hat{\mathbf{p}}_R\end{cases} \tag{28}$$

Hence, if we take

$$\begin{cases}A_L = r_0 e^{-\mathrm{i}\gamma_L r_0}\dfrac{4\pi\gamma^2}{k\gamma_L}\\ A_R = r_0 e^{-\mathrm{i}\gamma_R r_0}\dfrac{4\pi\gamma^2}{k\gamma_R}\end{cases} \tag{29}$$

then

$$\lim_{r_0\to\infty}\mathbf{Q}^{inc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_j) = e^{-\mathrm{i}\gamma_L\hat{\mathbf{r}}_0\cdot\mathbf{r}}\hat{\mathbf{p}}_j = = \mathbf{Q}^{inc}_j(\mathbf{r}; -\hat{\mathbf{r}}_0, \hat{\mathbf{p}}_j) \tag{30}$$

It is convenient to write the incident spherical Beltrami fields as [6], [7],

$$\mathbf{Q}^{inc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L) = \tilde{\mathbf{\Gamma}}_L(\frac{h(\gamma_L u)}{h(\gamma_L r_0)})\cdot\hat{\mathbf{p}}_L \tag{31}$$

$$\mathbf{Q}^{inc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_R) = \tilde{\mathbf{\Gamma}}_R(\frac{h(\gamma_R u)}{h(\gamma_R r_0)})\cdot\hat{\mathbf{p}}_R \tag{32}$$

where

$$\tilde{\mathbf{\Gamma}}_L = \frac{1}{2\gamma_L}(\gamma_L\tilde{\mathbf{I}} + \frac{1}{\gamma_L}\nabla\nabla + \nabla\times\tilde{\mathbf{I}}) \tag{33}$$

$$\tilde{\mathbf{\Gamma}}_R = \frac{1}{2\gamma_R}(\gamma_R\tilde{\mathbf{I}} + \frac{1}{\gamma_R}\nabla\nabla - \nabla\times\tilde{\mathbf{I}}) \tag{34}$$

and $u = |\mathbf{r} - \mathbf{r}_0|$. Using asymptotic forms (24) for $r\to\infty$ we obtain [6], [7],

$$\mathbf{Q}^{inc}_{j,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_j) = \mathbf{F}^{inc}_{j,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_j)h(\gamma_j r) + + O(\frac{1}{r^2}) \tag{35}$$

where

$$\mathbf{F}^{inc}_{j,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_j) = \frac{e^{-\mathrm{i}\gamma_j\hat{\mathbf{r}}\cdot\mathbf{r}_0}}{h(\gamma_j r_0)}\tilde{\mathbf{K}}_1(\hat{\mathbf{r}})\cdot\hat{\mathbf{p}}_j \tag{36}$$

are the far-field patterns of the point source incident Beltrami fields, which satisfy the relations

$$\hat{\mathbf{r}}\cdot\mathbf{F}^{inc}_{j,\mathbf{r}_0}(\hat{\mathbf{r}}\,|\,\hat{\mathbf{p}}_j) = 0 \tag{37}$$

The scattered electric field $\mathbf{E}^{sc}_{\mathbf{r}_0}$ will be dependent on the polarization $\hat{\mathbf{p}}_L$, $\hat{\mathbf{p}}_R$ and will be have the decomposition

$$\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) = \mathbf{Q}^{sc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) + \mathbf{Q}^{sc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R) \tag{38}$$

where $\mathbf{Q}^{sc}_{L,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ and $\mathbf{Q}^{sc}_{R,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ are the corresponding scattered Beltrami fields, which have the following behavior, when $r\to\infty$ [1], [10], [12]

$$\mathbf{Q}^{sc}_{j,\mathbf{r}_0}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)=h(\gamma_j r)\mathbf{g}_{j,\mathbf{r}_0}(\hat{\mathbf{r}})+ + O(\frac{1}{r^2}) \tag{39}$$

The functions $\mathbf{g}_{j,\mathbf{r}_0}(\hat{\mathbf{r}})$ given by

$$\mathbf{g}_{L,\mathbf{r}_0}(\hat{\mathbf{r}})=\frac{\mathrm{i}k\gamma_L}{8\pi\gamma^2}\tilde{\mathbf{K}}_L(\hat{\mathbf{r}})\cdot\int_S \hat{\mathbf{n}}\times(\gamma_L\nabla\times\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}')+\gamma^2\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}'))e^{-\mathrm{i}\gamma_L\hat{\mathbf{r}}\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{40}$$

$$\mathbf{g}_{R,\mathbf{r}_0}(\hat{\mathbf{r}})=\frac{\mathrm{i}k\gamma_R}{8\pi\gamma^2}\tilde{\mathbf{K}}_R(\hat{\mathbf{r}})\cdot\int_S \hat{\mathbf{n}}\times(\gamma_R\nabla\times\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}')-\gamma^2\mathbf{E}^{sc}_{\mathbf{r}_0}(\mathbf{r}'))e^{-\mathrm{i}\gamma_R\hat{\mathbf{r}}\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{41}$$

are the LCP and RCP far-field patterns, respectively [10], and they are dependent also on the polarization $\hat{\mathbf{p}}_L$ and $\hat{\mathbf{p}}_R$. We note that the far-field patterns $\mathbf{g}_{j,\mathbf{r}_0}(\hat{\mathbf{r}})$ satisfy the relations

$$\begin{cases}\hat{\mathbf{r}}\cdot\mathbf{g}_{j,\mathbf{r}_0}(\hat{\mathbf{r}})=0\\ \hat{\mathbf{r}}\times\mathbf{g}_{L,\mathbf{r}_0}(\hat{\mathbf{r}})=-\mathrm{i}\mathbf{g}_{L,\mathbf{r}_0}(\hat{\mathbf{r}})\\ \hat{\mathbf{r}}\times\mathbf{g}_{R,\mathbf{r}_0}(\hat{\mathbf{r}})=\mathrm{i}\mathbf{g}_{R,\mathbf{r}_0}(\hat{\mathbf{r}})\end{cases} \tag{42}$$

Reciprocity and General Scattering Theorems

For two vector functions $\mathbf{u}$ and $\mathbf{v}$ we introduce the bilinear form [10],

$$\{\mathbf{u},\mathbf{v}\}_S= =\int_S \hat{\mathbf{n}}\cdot(\mathbf{u}\times\nabla\times\mathbf{v}-\mathbf{v}\times\nabla\times\mathbf{u})dS- -2\beta\gamma^2\int_S \hat{\mathbf{n}}\cdot(\mathbf{u}\times\mathbf{v})dS \tag{43}$$

where S is the surface of the scatterer Ω^- and $\hat{\mathbf{n}}$ is the outward unit vector on S. We, also, consider two locations for the point source, **a** and **b** from which the time-harmonic incident spherical electric waves

$$\mathbf{E}^{inc}_{\boldsymbol{\sigma}}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)= =\mathbf{Q}^{inc}_{L,\boldsymbol{\sigma}}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L)+\mathbf{Q}^{inc}_{R,\boldsymbol{\sigma}}(\mathbf{r}\,|\,\hat{\mathbf{p}}_R) \tag{44}$$

for $\boldsymbol{\sigma}=\mathbf{a},\mathbf{b}$, emanate. $\mathbf{E}^{sc}_{\boldsymbol{\sigma}}$ and the corresponding scattered electric waves $\mathbf{E}^{sc}_{\boldsymbol{\sigma}}$ of form (38), have the Bohren decomposition of (9) in terms of LCP and RCP Beltrami fields. In particular, we have

$$\begin{cases}\mathbf{E}^{inc}_{\boldsymbol{\sigma}}=\mathbf{Q}^{inc}_{L,\boldsymbol{\sigma}}+\mathbf{Q}^{inc}_{R,\boldsymbol{\sigma}}\\ \mathbf{E}^{sc}_{\boldsymbol{\sigma}}=\mathbf{Q}^{sc}_{L,\boldsymbol{\sigma}}+\mathbf{Q}^{sc}_{R,\boldsymbol{\sigma}}\end{cases} \tag{45}$$

and the following properties for the fields $\mathbf{Q}^{inc}_{j,\boldsymbol{\sigma}}$ and $\mathbf{Q}^{sc}_{j,\boldsymbol{\sigma}}$ [6], [7]:

$$\{\bar{\mathbf{Q}}^{inc}_{L,\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}_L),\mathbf{Q}^{sc}_{L,\mathbf{b}}(\mathbf{r}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\}_S=\frac{2\gamma^2}{k}\int_S\hat{\mathbf{n}}\cdot(\bar{\mathbf{Q}}^{inc}_{L,\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}_L)\times\mathbf{Q}^{sc}_{L,\mathbf{b}}(\mathbf{r}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R))ds(\mathbf{r}) \quad (46)$$

$$\{\bar{\mathbf{Q}}^{inc}_{L,\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}_L),\mathbf{Q}^{sc}_{R,\mathbf{b}}(\mathbf{r}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\}_S=\mathbf{0} \quad (47)$$

$$\{\bar{\mathbf{Q}}^{inc}_{R,\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}_R),\mathbf{Q}^{sc}_{R,\mathbf{b}}(\mathbf{r}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\}_S=-\frac{2\gamma^2}{k}\int_S\hat{\mathbf{n}}\cdot(\bar{\mathbf{Q}}^{inc}_{R,\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}_R)\times\mathbf{Q}^{sc}_{R,\mathbf{b}}(\mathbf{r}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R))ds(\mathbf{r}) \quad (48)$$

$$\{\bar{\mathbf{Q}}^{inc}_{R,\mathbf{a}}(\mathbf{r};\hat{\mathbf{p}}_R),\mathbf{Q}^{sc}_{L,\mathbf{b}}(\mathbf{r}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\}_S=\mathbf{0} \quad (49)$$

Now a "reciprocity theorem" for spherical electromagnetic waves in chiral media is formulated as follows [6], [7]:

For any two incident spherical electric waves $\mathbf{E}^{inc}_{\mathbf{a}}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ and $\mathbf{E}^{inc}_{\mathbf{b}}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ of (44) and for any scatterer in a homogeneous isotropic chiral medium, we have:

$$\begin{aligned}&\frac{ae^{-i\gamma_L a}}{\gamma_L}\mathbf{Q}^{sc}_{L,\mathbf{b}}(\mathbf{a}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\cdot\hat{\mathbf{p}}_L+\\&+\frac{ae^{-i\gamma_R a}}{\gamma_R}\mathbf{Q}^{sc}_{R,\mathbf{b}}(\mathbf{a}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\cdot\hat{\mathbf{p}}_R=\\&=\frac{be^{-i\gamma_L b}}{\gamma_L}\mathbf{Q}^{sc}_{L,\mathbf{a}}(\mathbf{b}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\cdot\hat{\mathbf{q}}_L+\\&+\frac{be^{-i\gamma_R b}}{\gamma_R}\mathbf{Q}^{sc}_{R,\mathbf{a}}(\mathbf{b}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\cdot\hat{\mathbf{q}}_R\end{aligned} \quad (50)$$

Let $\mathbf{E}^{inc}_{\mathbf{a}}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ and $\mathbf{E}^{inc}_{\mathbf{b}}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ be two incident spherical waves of the form (44). We define spherical far-field pattern generators for LCP and RCP spherical Beltrami fields by

$$\mathbf{G}_{j,\mathbf{b}}(\mathbf{a}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)=\frac{ae^{i\gamma_L a}}{i\gamma_j}\cdot$$

$$\cdot\{\frac{1}{4\pi}\int_{S^2}\mathbf{g}_{j,\mathbf{b}}(\mathbf{a}|\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\cdot\overline{\tilde{\mathbf{K}}_j(\hat{\mathbf{r}})}e^{i\gamma_A\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}})-$$

$$-\mathbf{Q}^{sc}_{j,\mathbf{b}}(\mathbf{a}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\} \quad (51)$$

for $j=L,R$.

This terminology and definition is appropriate because when both the observation point and the source goes to infinity, $\mathbf{G}_{j,\mathbf{b}}(\mathbf{a}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)$, reduce to the far-field patterns for an incident plane electric wave propagating in the direction $-\hat{\mathbf{b}}$ and of polarizations $\hat{\mathbf{q}}_L$ for LCP and $\hat{\mathbf{q}}_R$ for RCP fields. Far-field pattern generators in acoustic and achiral electromagnetic scattering have been defined in [13]. When the point sources are transferred at infinity, the generators $\mathbf{G}_{j,\mathbf{b}}(\mathbf{a}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)$ are transformed into far-field patterns scattering spherical electrical wave, namely [6], [7],

$$\lim_{a\to\infty}\mathbf{G}_{j,\mathbf{b}}(\mathbf{a}|\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)=\boldsymbol{g}_{j,\mathbf{b}}(-\hat{\mathbf{a}}|\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R) \quad (52)$$

Using this notation, the "general scattering theorem" for spherical electric waves in chiral media is formulated as follows [6], [7]:

Let $\mathbf{E}_{\mathbf{a}}^{inc}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ and $\mathbf{E}_{\mathbf{b}}^{inc}(\mathbf{r}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)$ be two spherical electric waves of (44) incident upon a scatterer in a chiral medium. Then it is valid

$$\mathbf{G}_{L,\mathbf{b}}(\mathbf{a}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\cdot\hat{\mathbf{p}}_L+\mathbf{G}_{R,\mathbf{b}}(\mathbf{a}\,|\,\hat{\mathbf{q}}_L,\hat{\mathbf{q}}_R)\cdot\hat{\mathbf{p}}_R+$$
$$+\overline{\mathbf{G}_{L,\mathbf{a}}(\mathbf{b}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)}\cdot\hat{\mathbf{q}}_L+\overline{\mathbf{G}_{R,\mathbf{a}}(\mathbf{b}\,|\,\hat{\mathbf{p}}_L,\hat{\mathbf{p}}_R)}\cdot\hat{\mathbf{q}}_R=$$
$$=-\frac{1}{2\pi}\{\int_{S^2}\frac{1}{\gamma_L^2}\overline{\mathbf{F}_{L,\mathbf{a}}^{sc}(\hat{\mathbf{r}})}\cdot\mathbf{F}_{L,\mathbf{b}}^{sc}(\hat{\mathbf{r}})dS(\hat{\mathbf{r}})+$$
$$+\int_{S^2}\frac{1}{\gamma_R^2}\overline{\mathbf{F}_{R,\mathbf{a}}^{sc}(\hat{\mathbf{r}})}\cdot\mathbf{F}_{R,\mathbf{b}}^{sc}(\hat{\mathbf{r}})dS(\hat{\mathbf{r}})\} \quad (53)$$

References

[1] Lakhtakia A., "Beltrami Fields in Chiral Media", World Scientific, Singapore, 1993, §1-1 (p. 3), §3-1 (p. 83), §3-2.2 (p. 110), §4-1 (p. 130), §4-3,(p. 150), §5-1, (p. 164)

[2] Athanasiadis C., Martin P.A., Stratis I.G., "On the scattering of spherical electromagnetic waves by a penetrable chiral obstacle", 6th International Workshop on Mathematical Methods in Scattering Theory and Biomedical Ehgineering ,Tsepelovo, Greece, 2003

[3] Charalambopoulos A., Dassios G., Kamvyssas G., "Reciprocity theorems for point source scalar scattering", In: Dassios G., Fotiadis D., Kiriaki C., Massalas C. (eds) Mathematical Methods in Scattering Theory and Biomedical Technology, Longman, London 1998, pp. 12-19

[4] Dassios G, Kamvyssas G. "The impedance scattering problem for a point source field. The small resistive sphere", Quarterly Journal of Mechanics and Applied Mathematics, Vol. 50, 1999, pp. 321-332

[5] Athanasiadis C., Martin P.A., Stratis I.G., "On the Scattering of Point-generated Electromagnetic Waves by a Perfectly Conducting Sphere, and Related Near-field Inverse Problems", Zeitschrift für Angewandte Mathematik und Mechanik, Vol. 83, Issue 2, 2003, pp. 129-136

[6] Athanasiadis C, Berketis N., "Scattering relations for point – source excitation in chiral media", Mathematical Methods in the Applied Sciences, Vol. 29, 2006, pp. 27-48

[7] Berketis M. Nikolaos, "Scattering spherical Electromagnetic waves in chiral media", PhD thesis, University of Athens – Department of Mathematics, December 2007
[8] Colton D., Kress R., "Inverse Acoustic and Electromagnetic Scattering Theory" Springer, Berlin, 1992, §6.2, (p. 153)
[9] Bohren C. F., "Light scattering by an optically active Sphere", Chemical Physics Letters, Vol. 29, Issue 3, 1974, pp. 458-462
[10] Athanasiadis C., "On the far field patterns for electromagnetic scattering by a chiral obstacle in a chiral environment", Journal of Mathematical Analysis and Applications, Vol. 309, 2005, pp. 517-533
[11] Athanasiadis C., Giotopoulos S., "Scattering theorem for dyadic chiral fields", Reports on Mathematical Physics, Vol. 53, 2004, pp. 339-350
[12] Berketis M. Nikolaos, "Direct and Related Far-Field Inverse Scattering Problems for Spherical Electromagnetic Waves in Chiral Media", FunkTechnikPlus # Journal, Issue 2, November 2013, pp. 7-18
[13] Athanasiadis C., Martin P.A., Spyropoylos A., Stratis I.G., "Scattering relations for point source acoustic and electromagnetic waves", Journal of Mathematical Physics Vol. 43, Issue 11, 2002, pp. 5683-5697

Previous Publication in FUNKTECHNIKPLUS # JOURNAL

"Near-field Method in Solving Inverse Scattering Problem of Spherical Electromagnetic Waves in Chiral media",
Issue 3, pp. 19-25

*** About the Author**

Nikolaos Berketis, Issue 2, p. 18